GEODESICS, GEODESIC CURVATURE, GEODESIC PARALLELS, GEODESIC COORDINATES, GEODESIC TORSION, GAUSS-BONNET THEOREM

Geodesic. Intuitively, a geodesic is the shortest arc between two points on a surface. If we stretch a rubber band between two points on a convex surface, the rubber band will take the path of the geodesic. See Fig. 1. A geodesic C on a surface S has the properties that at each point of C the principal normal coincides with the normal to S and the geodesic curvature vanishes identically. Conversely, if on a curve C on a surface S, the principal normal coincides with the surface normal at every point, or if the geodesic curvature vanishes identically at every point, the curve is a geodesic. If a straight line lies on a surface, then the line is a geodesic of the surface.

Although defining a geodesic as the shortest arc between two points on a surface gives the main idea of a geodesic, there is a problem with it as a definition. Not every geodesic is a shortest path in the large, as can be seen by noting that on the surface of a sphere every arc of a great circle is a geodesic even though an arc will be the shortest path between two points only if that arc is not greater than a semicircle. From this example we see that a geodesic can be a closed curve. Because of this difficulty a geodesic is often defined as an arc C on a surface S at each point of which the principal normal coincides with the normal to S — or an arc at every point of which the geodesic curvature vanishes identically.

Theorem 1. If two surfaces are tangent along a curve that is a geodesic on one of them, then this curve will be a geodesic on the other.

Reason? At each point along the curve the principal normals coincide with the surface normals.

For an illustration of this theorem see Fig. 2. A straight plane strip of paper with a median line drawn on it is placed in different positions on a cylinder. One can see from this example that a geodesic line on a cylinder can be one of three things: a straight line, a circle or a helix.

Geodesic curvature. Let C be a curve on a surface S. The
geodesic curvature of C at a given point P is defined as the curvature, at P, of the orthogonal projection of C onto the plane Q tangent to S at point P. See Fig. 3, where C* is the projection of C onto the tangent plane Q. The geodesic curvature of C at P is defined then as the curvature of C* at P.

For the motivation that led to this definition let us note that if C represents an arc of minimum length between two points A and B on a surface S and point P lies on the arc then the projection of C onto the tangent plane at point P should be a straight line and the geodesic curvature of C in that situation should be counted as zero. Thus as candidates for the arcs of minimum length we are led to consider those curves where the curvature of the orthogonal projection of the curve onto the tangent plane is zero.

**Formula for computing geodesic curvature.** Let the curvature of curve C at point P be given by the vector \( k \). Let \( N \) be the unit surface normal at point P, \( T \) be the unit tangent vector to C at point P and \( U \) be a unit vector in the tangent plane Q defined by \( U = N \times T \) creating the orthonormal triad shown in Fig. 4. Vector \( k \) is orthogonal to \( T \), lying in the \( N-U \) plane. Then the geodesic curvature vector \( k_g \) is the component of \( k \) along vector \( U \) i.e.

\[
k_g = (k \cdot U) U
\]

**Theorem 2.** The geodesic curvature vector \( k_g \) of a curve C at P is the vector projection of the curvature vector \( k \) of C at P onto the tangent plane at P.

One can write the vector \( k \) as

\[
k = k_g + k_n
\]

where \( k_g \) is the component of \( k \) along the \( U \) axis and \( k_n \) is its component along the \( N \) axis. The component \( k_n \) along the \( N \) axis is the normal curvature of the surface S at P in the direction of tangent \( T \) i.e. the curvature of the normal section defined by \( N \) and \( T \).

Let the magnitude of the vector \( k_g \) be denoted by \( k_g \) i.e. \( k_g = |k_g| \). Then

\[
k_g = k_g U.
\]

The scalar \( k_g \) is called the **geodesic curvature** of C at P.

Let us denote the magnitude of the vector \( k \) by \( k \), i.e. \( k = |k| \). Then the curvature \( k \) of curve C at point P is related to the geodesic curvature \( k_g \) at P by

\[
k_g = k \cos \theta
\]
where $\theta$ is the angle between the osculating plane of $C$ and the tangent plane $Q$.

**Theorem 3.** If point $P$ on curve $C$ of surface $S$ is represented by the position vector $\vec{r}$ then $k_g$ is given by the following box products

$$k_g = [T k N]$$

or

$$k_g = \begin{bmatrix} \frac{d\vec{r}}{ds} & \frac{d^2\vec{r}}{ds^2} & N \end{bmatrix}$$

where $N$ is the normal to the surface at $P$, $T$ is the unit tangent to $C$ at $P$ and the over-dots represent differentiation with respect to curve length $s$.

This theorem follows from $k_g = k \cdot U = k \cdot (N \times T)$.

**Liouville’s formula for geodesic curvature.** The curves $C_1$ and $C_2$ of an orthogonal system on a surface are so directed that at each point the directed angle from the directed curve $C_1$ to the directed curve $C_2$ is $\pi/2$ (e.g. curves $C_1$ and $C_2$ could be $u$- and $v$-coordinate curves of an orthogonal system). The geodesic curvature of an arbitrary directed curve $C$: $u = u(s)$, $v = v(s)$ is then given by the formula

$$k_g = k_1 \cos \theta + k_2 \sin \theta + d\theta/ds$$

where $\theta$ is the directed angle at an arbitrary point $P$ of $C$ from the directed curve $C_1$ through $P$ to the directed curve $C$, $k_1$ is the geodesic curvature of curve $C_1$, $k_2$ is the geodesic curvature of curve $C_2$, and $s$ is arc length.

**Geodesics and asymptotic lines.** A curve is an asymptotic line if and only if at each point of the curve $k_n = 0$; a curve is a geodesic if and only if at each point of the curve $k_g = 0$. An asymptotic line is either a straight line or a curve along which the osculating plane and the tangent plane coincide; a geodesic is either a straight line or a curve along which the osculating plane is perpendicular to the tangent plane.

**Theorem 4.** The geodesic curvature $k_g$ of a curve is equal numerically to the ordinary curvature $k$ at every point of the curve if and only if the curve is an asymptotic line.

The geodesic curvature $k_g$ of a curve which is not a straight line is equal numerically to the ordinary curvature $k$ at every point of the curve if and only if the osculating plane of the curve at each point is the tangent plane to the surface at the point. As an example consider a circle in a plane, which meets the requirements for being an asymptotic line whose osculating plane at each point is the tangent plane.
Beltrami's formula for geodesic curvature. Given a curve C: \( u = u(s), v = v(s) \) on a surface \( S: \vec{r} = \vec{r}(u,v) \) where \( s \) is arc length. Beltrami's formula for the geodesic curvature at point P of the curve is:

\[
1) \quad k_g = \left[ \Gamma^2_{11} \left( \frac{du}{ds} \right)^3 + \left( 2 \Gamma^2_{12} - \Gamma^1_{11} \right) \frac{du}{ds} \frac{dv}{ds} + \left( \Gamma^2_{22} - 2 \Gamma^1_{12} \right) \frac{du}{ds} \left( \frac{dv}{ds} \right)^2 \right. \\
\left. - \Gamma^1_{22} \left( \frac{dv}{ds} \right)^3 + \frac{du}{ds} \frac{d^2 v}{ds^2} - \frac{d^2 u}{ds^2} \frac{dv}{ds} \right] \sqrt{EG - F^2}
\]

where the \( \Gamma^k_{ij} \) are the Christoffel symbols of the second kind.

**Derivation.**

In contrast to \( k_n \) which depends on both the first and second fundamental coefficients, the geodesic curvature \( k_g \) depends only on the first fundamental coefficients \( E, F \) and \( G \) (and their derivatives) and is thus an intrinsic property of the surface. This is evident in the above formula since the functions \( \Gamma^k_{ij} \) are functions of the first fundamental coefficients \( E, F \) and \( G \) only.

**Geodesic curvature of the coordinate curves.** We can compute the values of \( \frac{du}{ds} \) and \( \frac{dv}{ds} \) in 1) above for the \( u- \) and \( v- \)coordinate curves from the first fundamental quadratic form. For the \( u- \)coordinate curves where \( v = \) constant, \( \frac{dv}{ds} = 0 \) and \( \frac{du}{ds} = 1/ \sqrt{E} \). Along the \( v- \)coordinate curves where \( u = \) constant, \( \frac{du}{ds} = 0 \) and \( \frac{dv}{ds} = 1/ \sqrt{G} \). Substituting into 1) we get

\[
2) \quad \left( k_g \right)_{v = \text{constant}} = \Gamma^2_{11} \left( \frac{du}{ds} \right)^3 \sqrt{EG - F^2} = \Gamma^2_{11} \frac{\sqrt{EG - F^2}}{E \sqrt{E}} \\
\left( k_g \right)_{u = \text{constant}} = \Gamma^1_{22} \left( \frac{dv}{ds} \right)^3 \sqrt{EG - F^2} = -\Gamma^1_{22} \frac{\sqrt{EG - F^2}}{G \sqrt{G}}
\]

For the case where the coordinate curves are orthogonal, \( F = 0, \Gamma^2_{11} = -\frac{1}{2} E \sqrt{G}, \Gamma^1_{22} = -\frac{1}{2} G \sqrt{E} \). Thus
Theorem 5. The u-curves on a surface are geodesics if and only if $\Gamma_{11}^2 = 0$; and the v-curves if and only if $\Gamma_{22}^1 = 0$.

Proof. This theorem follows directly from 2) above.

Theorem 6. If the coordinate curves form an orthogonal system, the u-curves are geodesics when and only when $E$ is a function of $u$ alone, and the v-curves are geodesics when and only when $G$ is a function of $v$ alone.

Proof. This theorem follows directly from 3) above. For the u-curves, if $E$ is a function of $u$ alone, then $E_v = 0$ and $k_g = 0$. For the v-curves, if $G$ is a function of $v$ alone, then $G_u = 0$ and $k_g = 0$.

Corollary. If both the u-curves and the v-curves are geodesics then $E$ and $G$ are, respectively, functions $U(u)$ and $V(v)$, of $u$ and $v$.

Theorem 7. If there exists on a surface an orthogonal system of geodesics, the system is a developable or a plane.

Examples of systems of orthogonal geodesics.

1. Let $\alpha$ be the set of all lines in the plane directed in any specified direction. Let $\beta$ be the set of all lines in the plane perpendicular to the $\alpha$ lines. See Fig. 5. Then the two families of lines constitute an orthogonal system of geodesics.

2. Let $\alpha$ be the set of all lines in the surface of a cylinder parallel to the axis of revolution. Let $\beta$ be the set of all circles formed as plane sections by planes cutting through the cylinder perpendicular to the axis of revolution. See Fig. 6. Then the two families of lines constitute an orthogonal system of geodesics.
**Differential equations of the geodesics.**

**Theorem 8.** Let $S$ be a simple surface element defined by the one-to-one mapping

$$
\begin{align*}
x &= x(u, v) \\
y &= y(u, v) \\
z &= z(u, v)
\end{align*}
$$

of a region $R$ of the $uv$-plane into $xyz$-space. Then a directed curve $C$ on $S$ represented parametrically in terms of arc length $s$ by

$$
\begin{align*}
u &= u(s) \\
v &= v(s)
\end{align*}
$$

is a geodesic if and only if $u(s)$ and $v(s)$ satisfy the following differential equations:

4) $\frac{d^2 u}{ds^2} + \Gamma^1_{11} \left( \frac{du}{ds} \right)^2 + 2\Gamma^1_{12} \frac{du}{ds} \frac{dv}{ds} + \Gamma^1_{22} \left( \frac{dv}{ds} \right)^2 = 0$

$$
\frac{d^2 v}{ds^2} + \Gamma^2_{11} \left( \frac{du}{ds} \right)^2 + 2\Gamma^2_{12} \frac{du}{ds} \frac{dv}{ds} + \Gamma^2_{22} \left( \frac{dv}{ds} \right)^2 = 0
$$

Using equations 4) and an existence theorem from the theory of differential equations one can prove the following:

**Theorem 9.** In the neighborhood of a point $P$ on a surface of class $\geq 3$ there exists one and only one geodesic through $P$ in any given direction.

**Theorem 10.** Let $S$: $\vec{r} = \vec{r}(u, v)$ be a simple surface element of class $\geq 2$ such that $E = E(u)$, $F = 0$, and $G = G(u)$. Then

1. The $v$-curves are geodesics.
2. The $u$-curve $v = v_0$ is a geodesic if and only if $G_u(u_0) = 0$.
3. A curve of the form $\vec{r} = \vec{r}(u, v(u))$ is a geodesic if and only if

$$
v = \pm \int \frac{C\sqrt{E}}{\sqrt{G\sqrt{G - C^2}}} \, du
$$
where C is a constant.

Geodesic parallels, geodesic coordinates

**Def. Trajectory.** A curve which cuts all curves (or surfaces) of a given family at the same angle. An **orthogonal trajectory** is a curve which cuts all the members of a given family of curves (or surfaces) at right angles.

**James/James. Mathematics Dictionary.**

**Geodesic parallels.** The orthogonal trajectories of a family of lines in the plane have the following property: the segments cut from the lines by any two of them are all equal. The orthogonal trajectories of a family of geodesics on an arbitrary surface have the same property. Any two of them cut equal segments from the geodesics. Conversely, if any two orthogonal trajectories of a family of curves on a surface cut equal segments from the curves of the family, the curves of the family are geodesics on the surface.

**Theorem 11.** A necessary and sufficient condition that the segments cut from the curves of a family of curves on a surface by two arbitrarily chosen orthogonal trajectories of the family be all equal is that the curves of the family be geodesics on the surface.

**Def. Geodesic parallels.** The orthogonal trajectories of a family of geodesics are known as **geodesic parallels.** They are called parallels because each two of them are equally distant and, geodesic parallels because the distances in question are measured along geodesics.

**Example of geodesic parallels.** An example of geodesic parallels are the parallel circles that are traced out on a surface of revolution by the individual points of the plane curve that is rotated to generate the surface. The parallels of latitude on a sphere are geodesic parallels.

By Theorem 7, a family of geodesic parallels can consist of geodesics only if the surface is a developable or a plane. Normally geodesic parallels are not geodesics.

**Geodesic parallels on a surface.** Given a smooth curve $C_0$ on a surface $S$, there exists a unique family of geodesics on $S$ intersecting $C_0$ orthogonally. If segments of equal length $s$ be measured along the geodesics from $C_0$, then the locus of their end points is an orthogonal trajectory $C_s$ of the geodesics. The curves $C_s$ are **geodesic parallels.**

**James/James. Mathematics Dictionary.**

**Geodesic coordinates.** A simple surface element where the coordinate curves are orthogonal and one of the families of coordinate curves are geodesics is called a **set of geodesic coordinates.**
**Def. Geodesic parameters (coordinates).** Parameters \( u, v \) for a surface \( S \) such that the curves \( u = \text{const.} \) are the members of a family of geodesic parallels, while the curves \( v = \text{const.} = v_0 \) are members of the corresponding orthogonal family of geodesics, of length \( u_2 - u_1 \) between the points \((u_1, v_0)\) and \((u_2, v_0)\). A necessary and sufficient condition that \( u, v \) be geodesic parameters is that the first fundamental form of \( S \) reduce to \( ds^2 = du^2 + G dv^2 \).

From Theorem 6 we know that a necessary and sufficient condition for the \( u \)-curves to be geodesics and the \( v \)-curves to be geodesic parallels orthogonal to them is that \( E = U(u) \) and \( F = 0 \). This means that the first fundamental form has the form

\[
5) \quad ds^2 = U(u)du^2 + Gdv^2.
\]

If we then make a substitution

\[
\begin{align*}
d\bar{u} &= \sqrt{U(u)} \ du \\
\bar{v} &= v
\end{align*}
\]

5) becomes

\[
ds^2 = d\bar{u}^2 + Gd\bar{v}^2
\]

**Geodesic polar coordinates.** These are geodesic parameters \( u, v \) for a surface \( S \), except that the curves \( u = \text{const.} = u_0 \), instead of being geodesic parallels, are concentric geodesic circles, of radius \( u_0 \), and center, or pole, \( P \) corresponding to \( u = 0 \); the curves \( v = v_0 \) are the geodesic radii; and for each \( v_0 \), \( v_0 \) is the angle at \( P \) between the tangents to \( v = 0 \) and \( v = v_0 \). Necessary and sufficient conditions that \( u, v \) be geodesic polar coordinates are that the first fundamental quadratic form of \( S \) reduce to \( ds^2 = du^2 + \mu^2 dv^2 \), \( \eta \geq 0 \), and that at \( u = 0 \) we have \( \eta = 0 \) and \( \partial \eta / \partial u = 1 \). All points on \( u = 0 \) are singular points corresponding to \( P \).

**Geodesic circle on a surface.** If equal lengths are laid off from a point \( P \) of a surface \( S \) along the geodesics through \( P \) on \( S \), the locus of the end points is an orthogonal trajectory of the geodesics. The locus of end points is called a **geodesic circle** with center at \( P \) and radius \( r \). The “radius” \( r \) is a **geodesic radius**; it is the geodesic distance on the surface from the “center” \( P \) to the circle. See “geodesic polar coordinates”.

Bonnet’s formula for geodesic curvature. Let a curve $C$ on a surface $S$: $\mathbf{r} = \mathbf{r}(u, v)$ be defined by an equation of the form

$$\phi(u, v) = \text{const.}$$

Then Bonnet’s formula for the geodesic curvature at point $P$ of the curve is:

$$k_g = \pm \frac{1}{\sqrt{EG - F^2}} \left[ \frac{\partial}{\partial u} \frac{F\phi_v - G\phi_u}{\sqrt{E\phi_v^2 - 2F\phi_v\phi_u + G\phi_u^2}} + \frac{\partial}{\partial v} \frac{F\phi_u - E\phi_v}{\sqrt{E\phi_v^2 - 2F\phi_v\phi_u + G\phi_u^2}} \right]$$

where $E$, $F$, $G$, $L$, $M$, $N$ are the first and second fundamental coefficients.

Geodesic torsion.

Torsion of a geodesic. The torsion of a geodesic passing through point $P$ of a surface in the direction of the unit tangent vector $\mathbf{T}$ is given by

$$7) \quad \tau_g = \frac{d\mathbf{N}}{ds} \times \frac{d\mathbf{r}}{ds} \cdot \mathbf{N},$$

where $\mathbf{N}$ is the unit surface normal at point $P$ and $\mathbf{r}$ is the position vector at point $P$, or, equivalently, by

$$8) \quad \tau_g = \frac{(EM - FL)du^2 + (EN - GL)dudv + (FN - GM)dv^2}{\sqrt{EG - F^2(Edu^2 + 2Fdudv + Gdv^2)}}$$

where $E$, $F$, $G$, $L$, $M$, $N$ are the first and second fundamental coefficients.

Derivation

We note that the condition that the geodesic torsion $\tau_g = 0$ along a curve is tantamount to the condition that the curve is a line of curvature. A curve is a line of curvature if and only if at each point on the curve the direction of its tangent satisfies

$$9) \quad (EM - FL)du^2 + (EN - GL)dudv + (FN - GM)dv^2 = 0.$$

The left member of 9) is identical to the numerator in 8) so if $\tau_g = 0$ along a curve the curve is a line of curvature.
Theorem 12. The geodesic torsion of a curve is identically zero if and only if the curve is a line of curvature.

Theorem 13. A geodesic, which is not a straight line, is a plane curve if and only if it is a line of curvature.

Def. Geodesic torsion of a surface at a point in a given direction. The expression for $\tau_g$ in 7) above is similar to the expression for normal curvature in that it depends only on the point $P$ and on the direction $dv/du$ at $P$. The quantity $\tau_g$ defined by 7) is called the geodesic torsion of the surface at $P$ in the direction $dv/du$. Though the torsion at $P$ of the geodesic which issues from $P$ in a given direction fails to exist when the geodesic is a straight line, the geodesic torsion always exists because 7) defines it for every direction at $P$, regardless of the nature of the geodesic in the direction.

Theorem 14. The geodesic torsion at a point $P$ on a surface is related to the normal curvatures $k_1$ and $k_2$ at $P$ by

$$10) \quad \tau_g = \frac{1}{2} (k_2 - k_1) \sin 2\theta$$

where $\theta$ is the angle, in the tangent plane, measured counterclockwise from the direction of minimum curvature $k_1$.

At an umbilic, the geodesic torsion is zero in every direction. If we exclude this case, 9) reveals that the geodesic torsion is zero in the two principal directions and takes on its extreme values in the two perpendicular directions that bisect the angles between the principal directions.

Since $\sin(2(\theta + \pi/2)) = -\sin 2\theta$ and $\sin 2(-\theta) = -\sin 2\theta$ we have the following:

Theorem 15. 1. The geodesic torsions in two perpendicular directions, $\theta$ and $\theta + \pi/2$, are negatives of each another. 2. The geodesic torsions in two directions $+\beta$ and $-\beta$, as measured from a principal direction, are negatives of each other.

In particular, the two extreme values of the geodesic torsion at a point are negatives of each another. Also, the geodesic torsions in the two asymptotic directions are negatives of one another.

Def. Geodesic torsion of a curve on a surface. By the geodesic torsion of a curve $C$ (whether curve $C$ is a geodesic or not) at a point $P$ is meant the geodesic torsion of the surface at $P$ in the direction which $C$ has at $P$. Thus all curves through $P$ that have the same direction at $P$ have the same geodesic torsion at $P$. 
Theorem 16. The relationship between the geodesic torsion $\tau_g$ of a curve $C : u = u(s), v = v(s)$ at a point $P$ on a surface $S$ and the ordinary torsion $\tau$ of $C$ at point $P$ is given by

11) $\tau_g = \frac{d\alpha}{ds} - \tau$

where $\alpha$ is the directed angle from the unit surface normal $N$ to the principal normal $n$ to $C$.

We note that the angle $\alpha$ represents the angle between the osculating plane of $C$ and the tangent plane to the surface.

Theorem 17. The ordinary torsion and the geodesic torsion of a curve $C$, which is not a straight line, are identically equal if and only if the osculating plane of $C$ makes a constant angle with the tangent plane to the surface.

It follows from 11) that if $\tau_g = 0$, then $\tau = 0$ when and only when $d\alpha/ds = 0$. In other words:

Theorem 18. A necessary and sufficient condition that a line of curvature, other than a straight line, be a plane curve is that its osculating plane always makes the same angle with the tangent plane to the surface.

The Gauss-Bonnet Theorem, published by Bonnet in 1848, is an application of Green’s theorem to the integral of geodesic curvature.

**Gauss-Bonnet Theorem.** If the Gaussian curvature $K$ of a surface is continuous in a simply connected region $R$ bounded by a closed curve $C$ composed of $k$ smooth arcs making at the vertices exterior angles $\theta_1, \theta_2, \ldots, \theta_k$, then

$$\int_C k_g \, ds + \iint_R K \, dA = 2\pi - \sum \theta_i \quad i = 1, 2, \ldots, k$$

where $k_g$ represents the geodesic curvature of the arcs. See Fig. 7.

References.